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NEW CASES OF ISOCONIC MOTIONS IN THE GENERALIZED PROBLEM OF THE DYNAMICS OF A RIGID BODY WITH A FIXED POINT†

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Conditions for the existence of isoconic motions in the generalized problem of the dynamics of a rigid body with a fixed point are studied in the case when the auxiliary variables are described by polynomial solutions of the types considered by Steklov [1], Goryachev [2] and Kowalewski [3]. Two new cases of isoconic solutions are determined.

THE HODOGRAPH method, which offers a direct kinematic interpretation of the motion of a rigid body with a fixed point based on Poinsot's theorem and Kharlamov's equations [4], has not only produced a new conception of the properties of such motion, but has also revealed the existence of whole new classes of motion. One of the most important is the class of isoconic motions, in which the moving and fixed hodographs of the angular velocity are symmetric about a plane tangent to them. These motions were first studied by Fabbri [5], who established their existence in Steklov's solution [1]. The isoconic property in Steklov's solution has also been derived by using hodographs. Cases of isoconic motions in the classical context of the motion of a rigid body were also observed in the treatments of Lagrange [7], Zhukovskii [8], Hess–Stretenskii [7] and Grioli [9]. In the generalized problem of the motion of a gyrostat with a fixed point, the only isoconic solutions to have been studied are those with a first layer (in Kharlamov's terminology) of corresponding invariant relation [10], and isoconic precession motions.‡

1. STATEMENT OF THE PROBLEM

Consider the motion of a charged, magnetized gyrostat with a fixed point in a field of potential and gyroscopic forces. The potential forces are due to the interaction of the magnets with the constant magnetic field, whose direction is represented by a unit vector $\boldsymbol{\nu}$, to the interaction of the electrical charges with the electric field, and to the Newtonian attraction of masses. The centres of the Newton and Coulomb attractive forces lie on an axis through the fixed point parallel to $\boldsymbol{\nu}$. The gyroscopic forces are produced by the Lorentz action of the magnetic field on the electrical charges moving in space (there are no currents in the gyrostat) and the cyclic motions of the rotors in the body-carrier.

The equations of motion of the problem may be written in vector notation as follows (see, e.g. [1]):

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$$A\boldsymbol{\omega}^{\cdot}=(A\boldsymbol{\omega}+\boldsymbol{\lambda})\times\boldsymbol{\omega}+\boldsymbol{\omega}\times B\boldsymbol{\nu}+\boldsymbol{\nu}\times(C\boldsymbol{\nu}-\mathbf{s})$$
$$\boldsymbol{\nu}^{\cdot}=\boldsymbol{\nu}\times\boldsymbol{\omega} \tag{1.1}$$

These equations have three first integrals

$$A\boldsymbol{\omega}\cdot\boldsymbol{\omega}-2(\mathbf{s}\cdot\boldsymbol{\nu})+C\boldsymbol{\nu}\cdot\boldsymbol{\nu}=2E,\quad \boldsymbol{\nu}\cdot\boldsymbol{\nu}=1$$
$$(A\boldsymbol{\omega}+\boldsymbol{\lambda})\cdot\boldsymbol{\nu}-\tfrac{1}{2}(B\boldsymbol{\nu}\cdot\boldsymbol{\nu})=k \tag{1.2}$$

where $\boldsymbol{\omega}$ is the angular velocity of the gyrostat, $\boldsymbol{\nu}$ is the unit vector pointing along the axis of symmetry of the force fields, $\boldsymbol{\lambda}$ is the gyrostatic moment, $\mathbf{s}$ is the vector of the generalized centre of mass, A is the inertia tensor of the gyrostat relative to the fixed point, and B and C are symmetric 3×3 matrices; dots over the variables denote relative differentiation.

A necessary and sufficient condition for the existence of isoconic motions in the generalized problem of rigid-body dynamics is that system (1.1) admits of an invariant relation [12]

$$\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\mathbf{c})=0 \tag{1.3}$$

where $\mathbf{c}$ is unit vector that remains fixed relative to the body-carrier.

Suppose that the matrices A, B and C in (1.1) and (1.2) are in diagonal form, $\boldsymbol{\omega}=(p,\ q,\ r)$, $\boldsymbol{\nu}=(\nu_1,\ \nu_2,\ \nu_3)$, $\mathbf{s}=(s,0,0)$, $\boldsymbol{\lambda}=(\lambda,\ 0,0)$. Then we obtain from (1.1) and (1.2)

$$A_1p^{\cdot}=(A_2-A_3)qr+B_3\nu_3q-B_2\nu_2r+(C_3-C_2)\nu_2\nu_3$$
$$A_2q^{\cdot}=(A_3-A_1)rp-\lambda r+B_1\nu_1r-B_3\nu_3p-s\nu_3+(C_1-C_3)\nu_3\nu_1 \tag{1.4}$$
$$A_3r^{\cdot}=(A_1-A_2)pq+\lambda q+B_2\nu_2p-B_1\nu_1q+s\nu_2+(C_2-C_1)\nu_1\nu_2$$
$$\nu_1^{\cdot}=r\nu_2-q\nu_3,\quad \nu_2^{\cdot}=p\nu_3-r\nu_1,\quad \nu_3^{\cdot}=q\nu_1-p\nu_2$$
$$A_1p^2+A_2q^2+A_3r^2-2s\nu_1+C_1\nu_1^2+C_2\nu_2^2+C_3\nu_3^2=2E$$
$$\nu_1^2+\nu_2^2+\nu_3^2=1$$
$$2(A_1p+\lambda)\nu_1+2A_2q\nu_2+2A_3r\nu_3-B_1\nu_1^2-B_2\nu_2^2-B_3\nu_3^2=2k$$

Let us assume that Eqs (1.4) have a solution in the following form

$$q^2=Q(p)=\sum_{k=0}^{n}b_kp^k,\quad r^2=R(p)=\sum_{i=0}^{m}c_ip^i$$
$$\nu_1=\varphi(p)=\sum_{j=0}^{l}a_jp^j,\quad \nu_2=q\psi(p),\quad \nu_3=r\kappa(p) \tag{1.5}$$
$$\psi(p)=\sum_{i=0}^{n_1}g_ip^i,\quad \kappa(p)=\sum_{j=0}^{m_1}f_jp^j$$

where n, m, n_1, m_1, l are natural numbers or zero, b_k, c_i, a_j, g_i, f_j are certain as yet undetermined parameters. In the classical problem of the motion of a rigid body with a fixed point, there are three known solutions with this structure [1–3]. Considering the conditions for the existence of quadratic invariant relations in the motion of a gyrostat in a gravitational field, Kharlamov [4] generalized Steklov's and Kowalewski's solutions and showed that Goryachev's solution cannot be generalized.

Substitute (1.5) into (1.4)

$$p^{\cdot}=(\varphi'(p))^{-1}(\psi(p)-\kappa(p))(Q(p)R(p))^{1/2} \tag{1.6}$$

$$(Q(p)\psi^2(p))'=2\varphi'(p)\psi(p)(p\kappa(p)-\varphi(p))(\psi(p)-\kappa(p))^{-1} \tag{1.7}$$

$$(R(p)\kappa^2(p))'=2\varphi'(p)\kappa(p)(\varphi(p)-p\kappa(p))(\psi(p)-\kappa(p))^{-1}$$

$$A_1(\psi(p)-\kappa(p))=\varphi'(p)[(C_3-C_2)\psi(p)\kappa(p)+B_3\kappa(p)-B_2\psi(p)+A_2-A_3]$$

$$A_2Q'(p)(\psi(p)-\kappa(p))=2\varphi'(p)[(C_1-C_3)\varphi(p)\kappa(p)-$$

$$-\kappa(p)(B_3p+s)+B_1\varphi(p)+(A_3-A_1)p-\lambda] \tag{1.8}$$

$$A_3R'(p)(\psi(p)-\kappa(p))=2\varphi'(p)[(C_2-C_1)\varphi(p)\psi(p)+$$

$$+\psi(p)(B_2p+s)-B_1\varphi(p)+(A_1-A_2)p+\lambda]$$

$$\varphi^2(p)+Q(p)\psi^2(p)+R(p)\kappa^2(p)-1=0$$

$$Q(p)(C_2\psi^2(p)+A_2)+R(p)(C_3\kappa^2(p)+A_3)+C_1\varphi^2(p)-$$

$$-2s\varphi(p)+A_1p^2-2E=0,\quad Q(p)\psi(p)(B_2\psi(p)-2A_2)+$$

$$+R(p)\kappa(p)[B_3\kappa(p)-2A_3]+B_1\varphi^2(p)-2A_1p\varphi(p)-2\varphi(p)\lambda+2k=0 \tag{1.9}$$

The prime denotes differentiation with respect to p. Equation (1.6) establishes the dependence of p on t. Note that Eqs (1.6) and (1.7) were obtained from the kinematic equations, Eqs (1.8) from the dynamical equations, and Eqs (1.9) from the integrals of the equations of motion.

In our examination of the conditions for the existence of isoconic motions of the gyrostat, we shall assume that the vector **c** in (1.3) points along the axis on which the gyrostat's centre of mass is situated. The class of isoconic motions that possess this property is not empty, since it can be shown that, for example, Steklov's solution [1] has the property. We then derive from (1.3), using (1.5)

$$p[\varphi(p)-c_1^*]+Q(p)\psi(p)+R(p)\kappa(p)=0 \tag{1.10}$$

where $c_1^*=\pm1$.

2. THE CASE $\psi(p)=g_0$, $\kappa(p)=f_0$

A preliminary task is to estimate the maximum degrees of the polynomials in (1.5), that is, the numbers n, m, n_1, m_1, l. As a first step, the estimate is conveniently derived from the first equation of system (1.8). There are several singular cases. Let $\psi(p)=g_0$, $\kappa(p)=f_0(n_1=0, m_1=0)$. Then, noting that $g_0-f_0\neq0$ (otherwise p=const and the gyrostat is rotating uniformly), we obtain from the equation

$$\varphi(p)=a_1p+a_0 \tag{2.1}$$

The first equation of system (1.9) and Eq. (1.10) yield

$$
\begin{aligned}
Q(p)\psi(p)[\psi(p)-\kappa(p)] &= p\kappa(p)[\varphi(p)-c_1^*]-\varphi^2(p)+1 \\
R(p)\kappa(p)[\psi(p)-\kappa(p)] &= p\psi(p)[c_1^*-\varphi(p)]+\varphi^2(p)-1
\end{aligned}
\tag{2.2}
$$

Thus, the maximum degrees of the polynomials $Q(p)$ and $R(p)$ do not exceed two

$$
Q(p)=b_2p^2+b_1p+b_0,\quad R(p)=c_2p^2+c_1p+c_0 \tag{2.3}
$$

The condition that (2.1) and (2.3) satisfy Eqs (1.7)–(1.9) gives a system of algebraic equations in the parameters, from which we obtain the following conditions

$$
\begin{aligned}
& b_0=0,\quad c_0=0,\quad a_0=c_1^* \\
& a_1=A_1(g_0-f_0)[(C_3-C_2)g_0f_0+B_3f_0-B_2g_0+A_2-A_3]^{-1} \\
& b_1=2a_0a_1g_0^{-1}(f_0-g_0)^{-1},\quad c_1=2a_0a_1f_0^{-1}(g_0-f_0)^{-1} \\
& b_2=a_1(f_0-a_1)g_0^{-1}(g_0-f_0)^{-1},\quad c_2=a_1(a_1-g_0)f_0^{-1}(g_0-f_0)^{-1} \\
& g_0(f_0s+\lambda)=a_0[g_0f_0(C_1-C_3)+A_2+g_0B_1] \\
& f_0(g_0s+\lambda)=a_0[g_0f_0(C_1-C_2)+A_3+f_0B_1] \\
& g_0f_0(C_3-C_2)[A_2f_0+g_0(A_1-A_3)+g_0f_0B_3]- \\
& -g_0f_0A_1(C_1-C_3)(g_0-f_0)=A_1(g_0-f_0)(A_2+g_0B_1)- \\
& -(B_3f_0-B_2g_0+A_2-A_3)[A_2f_0+g_0(A_1-A_3)+g_0f_0B_3] \\
& g_0f_0(C_3-C_2)[g_0(A_3-A_1)-f_0(A_2-2A_1)+g_0f_0B_2]- \\
& -g_0f_0A_1(C_1-C_3)(g_0-f_0)=A_1(g_0-f_0)(A_3+f_0B_1)- \\
& -(B_3f_0-B_2g_0+A_2-A_3)[A_3g_0-f_0(A_2-A_1)+g_0f_0B_2]
\end{aligned}
\tag{2.4}
$$

Naming g_0 and f_0 as free parameters, we can use the ninth and tenth equations of system (2.4) to determine s and λ, and the eleventh and twelfth equations to determine the quantities C_3-C_2, C_1-C_3. The parameters g_0 and f_0 must then satisfy the condition

$$
f_0g_0(B_3-B_2)+2f_0(A_2-A_1)-2g_0(A_3-A_1)\neq 0
$$

Let us consider an example in which the eleventh and twelfth equations of system (2.4) are solvable. Let $B_1=B_2=B_3=0$, $C_{ij}=\varepsilon^2A_{ij}$ (ε^2 being a parameter). The two equations mentioned then yield only one equation

$$
\varepsilon^2g_0f_0^2(A_1-A_2)+f_0[A_2-\varepsilon^2g_0^2(A_1-A_3)]-g_0A_3=0
$$

which can be solved for f_0, say, when $A_1-A_2>0$.

Thus, if $\psi(p)=g_0$, $\kappa(p)=f_0$ the solution (1.5) will be

$$
\begin{aligned}
& p^{\cdot}=(g_0-f_0)pa_1^{-1}[(b_2p+b_1)(c_2p+c_1)]^{1/2},\quad q^2=p(b_2p+b_1) \\
& r^2=p(c_2p+c_1),\quad \nu_1=a_1p+c_1^* \\
& \nu_2=g_0[p(b_2p+b_1)]^{1/2},\quad \nu_3=f_0[p(c_2p+c_1)]^{1/2}
\end{aligned}
\tag{2.5}
$$

Note that in the classical problem of motion of a heavy gyrostat there is no analogue for the solution (2.5) if $s \neq 0$. In addition, it follows from the form of the solution that, if $n_1 = 0$ and $m_1 = 0$, the polynomials $Q(p)$ and $R(p)$ cannot be constants, since $b_0 = 0$ and $c_0 = 0$.

The solution (2.5) will be real, for example, if $A_1 = 2a$, $A_2 = 1.5a$, $A_3 = a$, $g_0 = 2f_0$, $B_1 = B_2 = B_3 = 0$, $a_0 = 1$, where $a > 0$, $f_0 > 0$.

3. THE CASE $m_1 = 0$, $n_1 \neq 0$

The first equation of system (1.8) implies

$$A_1(\psi(p) - f_0) = \varphi'(p)\{\psi(p)[(C_3 - C_2)f_0 - B_2] + B_3 f_0 + A_2 - A_3\} \tag{3.1}$$

This yields two possible cases

$$(C_3 - C_2)f_0 - B_2 = 0, \quad n_1 = l - 1 \tag{3.2}$$

$$(C_3 - C_2)f_0 - B_2 \neq 0, \quad l = 1 \tag{3.3}$$

By (1.7)

$$\begin{aligned} &(Q(p)\psi^2(p))'(\psi(p) - f_0) = 2\psi(p)\varphi'(p)(pf_0 - \varphi(p)) \\ &R'(p)f_0(\psi(p) - f_0) = 2\varphi'(p)(\varphi(p) - p\psi(p)) \end{aligned} \tag{3.4}$$

Consider the case (3.2). Analysing (2.2) and (3.4), one can show that $n = 2$, $m \leqslant l + 1$. On the basis of (3.1), which takes the form

$$\psi(p) - f_0 = \mu_1\varphi'(p), \quad \mu_1 = A_1^{-1}(B_3 f_0 + A_2 - A_3) \tag{3.5}$$

Eqs (1.8) may be transformed as follows:

$$\begin{aligned} &\mu_1 A_2(2b_2 p + b_1) = 2\{\varphi(p)[(C_1 - C_3)f_0 + B_1] + \\ &+(A_3 - A_1 - B_3 f_0)p - (f_0 s + \lambda)\} \\ &\mu_1 A_3 R'(p) = 2\{(C_2 - C_1)\varphi(p)\psi(p) + \psi(p)(B_2 p + s) - B_1\varphi(p) + (A_1 - A_2)p + \lambda\} \end{aligned} \tag{3.6}$$

Since $l > 1$, the first equation of system (3.6) yields

$$\begin{aligned} &(C_1 - C_3)f_0 + B_1 = 0, \quad b_2 = \mu_2 / \mu_1, \quad b_1 = 2\mu_3 / \mu_1 \\ &\mu_2 = A_2^{-1}(A_3 - A_1 - B_3 f_0), \quad \mu_3 = -A_2^{-1}(f_0 s + \lambda) \end{aligned} \tag{3.7}$$

Since the maximum degree of the polynomial on the left of the second equation in (3.6) is at most l, and for $C_1 \neq C_2$ the right-hand side of that equation is a polynomial of maximum degree $2l - 1$, it necessarily follows that $C_2 = C_1$ in (3.6). It then follows from (3.2) and (3.7) that $B_2 = B_1$. Eliminating the quantity $\psi(p) - f_0$ in the second equation of (3.4) by using (3.5), we find that

$$R(p) = 2(f_0\mu_1)^{-1}[(a_l - g_{l-1})(l + 1)^{-1}p^{l+1} + \ldots + a_0 p + b_*] \tag{3.8}$$

where b_* is an arbitrary constant. Using (3.8) and the previously obtained conditions $C_2 = C_1$, $B_2 = B_1$, we can write down conditions under which (3.5) and the second equation of (3.6) are identified in p

$$
\begin{aligned}
&a_l = g_{l-1},\ a_{l-1} - g_{l-2} = \mu_4 g_{l-1},\dots,a_2 - g_1 = \mu_4 g_2 \\
&a_1 - g_0 = \mu_4 g_1 + \mu_5,\ \ a_0 = \mu_4 g_0 + \mu_6 \\
&l\mu_1 = 1,\ g_{l-2} = \mu_1(l-1)a_{l-1},\dots,g_1 = \mu_1 a_2,\ g_0 - f_0 = \mu_1 a_1 \\
&\mu_4 = sf_0(A_3 + B_3 f_0)^{-1},\ \ \mu_5 = f_0(A_1 - A_2)(A_3 + B_3 f_0)^{-1} \\
&\mu_6 = \lambda f_0(A_3 + B_3 f_0)^{-1}
\end{aligned} \tag{3.9}
$$

We have taken into account that $a_l \neq 0$ if $l > 1$. Consider Eq. (1.10)

$$p(\varphi(p) - c_1^*) + (b_2 p^2 + b_1 p + b_0)\psi(p) + f_0 R(p) = 0 \tag{3.10}$$

By (3.8) and (3.9), this implies that $b_2 = -1$, and then by (3.7) we have

$$(A_2 - A_1)(B_3 f_0 + A_1 + A_2 - A_3) = 0 \tag{3.11}$$

If $A_2 \neq A_1$ in (3.11), then $l = -1$, which is impossible. We therefore put $A_2 = A_1$ in (3.11). By (3.9)

$$a_{l-1} = l\mu_4 g_{l-1},\ \ g_{l-2} = (l-1)\mu_4 g_{l-1} \tag{3.12}$$

On the basis of these conditions, we require that the first equation in (3.4) and Eq. (3.10) must be identities with respect to p. We then obtain the following constraints on the parameters

$$\mu_1\mu_4(l-1)^2 - \mu_3(2l-1) - \mu_1\mu_4 l = 0,\ \ 3\mu_1\mu_4 + 2\mu_3 = 0$$

which cannot hold simultaneously if $l > 0$. This means that in (3.9) and (3.12) $g_{l-1} = 0$, $a_l = 0$ if $l > 1$. Consequently, case (3.2) cannot occur.

Consider case (3.3). It follows from (2.2) that $n = 0$, $n_1 = 1$, $m = 2$. If we require that at these parameter values the first equation of (1.9) and the first equation of (3.4) must be identities in p, we get

$$b_0 g_1^2 - a_1 f_0 + a_1^2 = 0,\ \ a_1^2 + b_0 g_1^2 + f_0^2 c_2 = 0,\ \ a_1 + b_0 g_1 + f_0 c_2 = 0$$

Since $b_0 \neq 0$ (otherwise $p = \text{const}$), it follows from these equalities that $g_1 = 0$ or $f_0 = 0$. Since $n_1 \neq 0$ it follows from (1.4) that the equality $f_0 = 0$ may be true provided that $q\nu_1 - p\nu_2 = 0$, which condition we reduce to the form $b_0\varphi^2(p) - p^2 R(p)\psi(p) = 0$. But this cannot hold identically in p if $n_1 \neq 0$.

We have thus shown that the case $n_1 \neq 0$, $m_1 = 0$ is impossible.

4. THE CASE $l = 1$ $n_1 \neq 0$, $m_1 \neq 0$

We may assume without loss of generality that $m_1 \leq n_1$. It then follows from the first equation of (1.8) that

$$C_3 = C_2 \tag{4.1}$$

Consider the first equation of (1.9) and Eq. (1.10)

$$
\begin{aligned}
&(a_1 p + a_0)^2 + Q(p)\psi^2(p) + R(p)\kappa^2(p) - 1 = 0 \\
&p[a_1 p + (a_0 - c_1^*)] + Q(p)\psi(p) + R(p)\kappa(p) = 0
\end{aligned} \tag{4.2}
$$

Analysing these equations, we obtain the following possible cases: if $n=0$, $n_1=2$: $m_1 \leq 0$; if $n=1$, $n_1=1$: $m_1=1$; if $n>1$: $m=n$, $m_1=n_1$. If $n=m=n_1=m_1=1$, it follows from the second equation of system (4.2) that $b_1g_1+c_1f_1=-a_1$. We then derive from (1.7) a system of algebraic equations, from which it follows that $3(b_1g_1+c_1f_1)=-2a_1$. This means that $a_1=0$, which is impossible.

We will now study the case when $n=m$, $n_1=m_1$, where $n_1 \neq 0$. Equations (4.2) imply the conditions

$$b_n g_{n_1}^2 + c_n f_{n_1}^2 = 0, \quad b_n g_{n_1} + c_n f_{n_1} = 0 \tag{4.3}$$

from which it follows that $g_{n_1}=f_{n_1}$, $b_n+c_n=0$. We now turn to the first equation in system (1.8). This equation, by virtue of our assumptions, yields $B_3=B_2$ and

$$\begin{gathered} g_{n-1} = f_{n_1-1}, \ldots, g_1 = f_1, \quad g_0 = f_0 + \mu_0 \\ \mu_0 = a_1(A_2 - A_3)(A_1 + a_1 B_2)^{-1} \end{gathered} \tag{4.4}$$

Here we will ignore the case $A_2=A_3$, for as $B_3=B_2$ (see (4.1)) and Eqs (1.4) are true, it leads to the Kirchhoff–Kharmalov solution.

By (4.3), the first equation of (2.2) becomes

$$\mu_0 Q(p)(\kappa(p)+\mu_0) = 1-(a_1p+a_0)^2 - p\kappa(p)[a_1p+(a_0-c_1^*)]$$

from which it follows that $n=2$. Then also $m=2$. If $a_0(C_1-C_2)-s \neq 0$, the second and third equations of systems (1.8) imply

$$\begin{aligned} & a_1(C_1-C_2)-B_2=0, \quad n_1=1 (f_{n_1}=0 \quad \text{if} \quad n_1>1) \\ & b_2\mu_0A_2 = a_1\{f_1[a_0(C_1-C_2)-s]+B_1a_1+A_3-A_1\} \\ & b_1\mu_0A_2 = 2a_1\{f_0[a_0(C_1-C_2)-s]+(B_2a_0-\lambda)\} \\ & c_2\mu_0A_3 = a_1\{f_1[a_0(C_2-C_1)+s]+A_1-A_2-B_1a_1\} \\ & c_1\mu_0A_3 = 2a_1\{f_0[a_0(C_2-C_1)+s]+a_0\mu_0(C_2-C_1)+\mu_0 s-B_1a_0+\lambda\} \end{aligned} \tag{4.5}$$

where, by (4.3) and (4.4), it must be true that $f_1(b_2+c_2)=0$. This leads to the equalities $b_2 = -a_1/\mu_0$, $c_2=a_1/\mu_0$.

By (4.2), we have

$$f_1(c_1+b_1)+2b_2\mu_0=0, \quad f_1(c_1+b_1)+b_2\mu_0+a_1=0$$

Taking everything into account, we obtain $a_1=0$, which is impossible. Hence we must put $a_0(C_1-C_2)-s=0$. Then it is true that $A_2b_1+A_3c_1=0$.

The second relationship in (4.2) may be written in the form

$$\begin{aligned} & p[a_1p+(a_0-c_1^*)]+(f_{n_1}p^{n_1}+\ldots+f_0)[(b_1+c_1)p+(b_0+c_0)]+ \\ & +\mu_0(b_2p^2+b_1p+b_0)=0 \end{aligned} \tag{4.6}$$

If $n_1=1$, one can use the previously obtained results, i.e. in that case $a_1=0$.

If $n_1 \neq 0$, it follows from (4.6) that $b_1+c_1=0$, so that $A_2=A_3$. Thus, the case $l=1$ is impossible.

5. THE GENERAL CASE ($l > 1,\ m_1 \neq 0,\ n_1 \neq 0$)

By virtue of our assumptions, condition (4.1) follows from the first equation of system (1.8). If it is also assumed that $n_1 > m_1$, it follows from that equation that $B_2 = 0$. We then deduce from (1.7) and (1.8) that

$$
\begin{aligned}
&A_1[\psi(p) - \kappa(p)] = \varphi'(p)[B_3\kappa(p) + A_2 - A_3] \\
&[Q(p)\psi^2(p)]' = 2A_1\psi(p)[p\kappa(p) - \varphi(p)][B_3\kappa(p) + A_2 - A_3]^{-1} \\
&[R(p)\kappa^2(p)]' = 2A_1\kappa(p)[\varphi(p) - p\psi(p)][B_3\kappa(p) + A_2 - A_3]^{-1} \\
&A_2Q'(p)[B_3\kappa(p) + A_2 - A_3] = 2A_1[(C_1 - C_2)\varphi(p)\psi(p) - \\
&-\kappa(p)(B_3p + s) + B_1\varphi(p) + (A_3 - A_1)p - \lambda] \\
&A_3R'(p)[B_3\kappa(p) + A_2 - A_3] = 2A_1[-(C_1 - C_2)\varphi(p)\kappa(p) + \\
&+\psi(p)s - B_1\varphi(p) + (A_1 - A_2)p + \lambda]
\end{aligned}
\tag{5.1}
$$

Since $n_1 > m_1$, it follows from the first equation of this system that the following cases are possible

$$
(1)\ B_3 = 0,\ n_1 = l - 1,\ (2)\ B_3 \neq 0,\ n_1 = m_1 + l - 1 \tag{5.2}
$$

Analysis of the other equations of (5.1) produces the following subcases

$$
\begin{aligned}
&1.1.\ C_1 \neq C_2,\ m + m_1 - 1 \leqslant l,\ n + n_1 - 1 = l,\ m - 1 = l - 1 + n_1 \\
&1.2.\ C_1 = C_2,\ m + m_1 - 1 \leqslant l,\ n - 1 \leqslant l,\ m - 1 = l \\
&2.1.\ C_1 \neq C_2,\ m + m_1 - 1 = l,\ n - 1 = l,\ m - 1 = l + n_1 - m_1 \\
&2.2.\ C_1 = C_2,\ m + m_1 - 1 = l,\ n + n_1 - 1 = \max(1, l - m_1)(l - m_1 \neq 1) \\
&n + n_1 - 1 \leqslant 1(l - m_1 = 1),\ n + m_1 - 1 = \max(m_1 + 1, l)(m_1 + 1 \neq l) \\
&n - 1 + m_1 \leqslant l(m_1 + 1 = l),\ m - 1 + m_1 = \max(n_1, l)(n_1 \neq l) \\
&m - 1 + m_1 \leqslant l\ (n_1 = l)
\end{aligned}
\tag{5.3}
$$

It can be shown that (5.2) and (5.3) cannot be simultaneously true. Thus, we must assume that $m_1 = n_1$.

By the first equation of system (1.9)

$$
\begin{aligned}
&\text{A1}.\ 2l = n + 2n_1 = m + 2m_1, \qquad \text{A2}.\ 2l = n + 2n_1 > m + 2m_1 \\
&\text{A3}.\ 2l = m + 2m_1 > n + 2n_1, \qquad \text{A4}.\ n + 2n_1 = m + 2m_1 > 2l
\end{aligned}
\tag{5.4}
$$

Similarly, we deduce from (1.10) the following independent possibilities

$$
\begin{aligned}
&\text{B1}.\ l + 1 = n + n_1 = m + m_1, \qquad \text{B2}.\ l + 1 = n + n_1 > m + m_1 \\
&\text{B3}.\ l + 1 = m + m_1 > n + n_1, \qquad \text{B4}.\ n + n_1 = m + m_1 > l + 1
\end{aligned}
\tag{5.5}
$$

Analysing the compatibility of relations (5.4) and (5.5), we obtain the following cases

$$\begin{aligned}
&1.\ n=m=2,\ m_1=n_1=l-1\\
&2.\ n=m=2(l-n_1),\ m_1=n_1<l-1\\
&3.\ n=2,\ m<2,\ m_1=n_1=l-1\\
&4.\ m=2,\ n<2,\ m_1=n_1=l-1\\
&5.\ m=n=l+1-n_1,\ m_1=n_1>l-1\\
&6.\ m=n>l+1-n_1,\ m=n>2(l-n_1)
\end{aligned} \tag{5.6}$$

Consider the first equation of (1.8)

$$\begin{aligned}
&A_1[(g_{n_1}-f_{n_1})p^{n}+\ldots+(g_0-f_0)]=\\
&=(la_lp^{l-1}+\ldots+a_1)[(f_{n_1}B_3-g_{n_1}B_2)p^{n_1}+\ldots+(f_1B_3-g_1B_2)p+(f_0B_3-g_0B_2+A_2-A_3)]
\end{aligned} \tag{5.7}$$

If we put $B_3=B_2$ in (5.7) and remember that $l>1$ we get $g_i-f_i=0$ for all $i=0,\ldots,n_1$. But then it follows from (1.6) that $p=\text{const}$, i.e. the gyrostat rotates uniformly. We shall therefore assume from now on that $B_3\neq B_2$. From (5.7) we now deduce the condition $g_{n_1}\neq f_{n_1}$ and the following bound for the maximal degrees $n_1\geqslant l-1$.

Let $n_1=l-1+N$, where N takes values $0,1,\ldots,n-1$. Then

$$\begin{aligned}
&f_{n_1}B_3-g_{n_1}B_2=0,\ldots,f_{N+1}B_3-g_{N+1}B_2=0\\
&f_NB_3-g_NB_2\neq 0\ (N\neq 0),\ f_0B_3-g_0B_2+A_2-A_3\neq 0\ (N=0)
\end{aligned} \tag{5.8}$$

By (5.7) and (5.8)

$$\begin{aligned}
&\psi(p)-\kappa(p)=A_1^{-1}(la_lp^{l-1}+\ldots+a_1)(\alpha_Np^N+\ldots+\alpha_0)\\
&\alpha_N=f_NB_3-g_NB_2,\ldots,\alpha_1=f_1B_3-g_1B_2,\ \alpha_0=f_0B_3-g_0B_2+A_2-A_3
\end{aligned} \tag{5.9}$$

Since $n_1=l-1+N$, $N\geqslant 0$, case 2 of (5.6) is impossible. Case 6 is impossible because of (2.2), from which it follows that $n\leqslant 2$ if $N=0$ and $n=2-N$ if $N\neq 0$. The second and third equations of (1.8) imply

$$\begin{aligned}
&A_2(nb_np^{n-1}+\ldots+b_1)[(g_{n_1}-f_{n_1})p^{n_1}+\ldots+(g_0-f_0)]=\\
&=2(la_lp^{l-1}+\ldots+a_1)[(C_1-C_2)a_lf_{n_1}p^{l+n_1}+\ldots]\\
&A_3(mc_mp^{m-1}+\ldots+c_1)[(g_{n_1}-f_{n_1})p^{n_1}+\ldots+(g_0-f_0)]=\\
&=2(la_lp^{l-1}+\ldots+a_1)[-(C_1-C_2)a_lg_{n_1}p^{l+n_1}+\ldots]
\end{aligned} \tag{5.10}$$

Hence, since $l>1$, $g_{n_1}\neq f_{n_1}$, it follows that $n=m=2l$. The last condition is incompatible with cases 1, 3, 4 and 5 of (5.6). Thus, we put $C_1=C_2=C_3$ in (5.10) and

$$\begin{aligned}
&A_2(nb_np^{n-1}+\ldots+b_1)[(g_{n_1}-f_{n_1})p^{n_1}+\ldots+(g_0-f_0)]=\\
&=2(la_lp^{l-1}+\ldots+a_1)[-(B_3p+s)(f_{n_1}p^{n_1}+\ldots+f_0)+\\
&+B_1(a_lp^l+\ldots+a_0)+(A_3-A_1)p-\lambda]
\end{aligned}$$

$$A_3(mc_m p^{m-1}+\ldots+c_1)[(g_{n_1}-f_{n_1})p^{n_1}+\ldots+(g_0-f_0)]= \tag{5.11}$$

$$=2(la_l p^{l-1}+\ldots+a_1)[-(B_2p+s)(g_{n_1}p^{n_1}+\ldots+g_0)-$$

$$-B_1(a_l p^l+\ldots+a_0)+(A_1-A_2)p+\lambda]$$

Let $N\neq 0$, i.e. $n_1>l-1$. Then, since $B_2\neq 0$, it follows from (5.11) that $n=l+1$, $m=l+1$. Hence case 5 of (5.6) is impossible.

We thus have only one remaining case in (5.6) $n_1=l-1$, $m\leqslant 2$, $n\geqslant 2$. And since $C_3=C_2=C_1$, the matrix C does not occur in the equation.

Examination of the energy integral in (1.9) gives $l=2$. Consequently, in the general case a solution of Eqs (1.7) and (1.8) can only in the case when $n=m=2$, $n_1=m_1=1$, $l=2$. Substituting these values into (1.5), we require that Eqs (1.7) and (1.8) must be identities in p. This gives the solution

$$q^2=a_2g_1^{-1}(g_1-f_1)^{-1}[(f_1-a_2)p^2-1]$$

$$r^2=a_2g_1^{-1}(g_1-f_1)^{-1}[(a_2-g_1)p^2+1],\quad \nu_1=a_2p^2+1 \tag{5.12}$$

$$\nu_2=g_1pq,\quad \nu_3=f_1pr,\quad p^{\cdot}=(g_1-f_1)(2a_2)^{-1}\sqrt{q^2r^2}$$

which exists under the following conditions

$$C_3=C_2=C_1,\quad a_2=A_1g_1(2A_2-A_1),\quad \lambda=B_1$$

$$f_1=g_1(A_1-2A_3)(A_1-2A_2)^{-1},\quad B_2=2\lambda A_1(2A_2-A_1)^{-1}$$

$$B_3=2\lambda A_1(2A_3-A_1)^{-1},\quad s=(A_2-A_1)(A_1-A_3)g_1^{-1}(A_1-2A_3)^{-1}$$

and generalizes Steklov's solution [1] to the motion of a gyrostat in a Lorentz field.

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